UNCLASSIFIED

AD 295 720

Reproduced
by the

ARMED SERVICES TECHNICAL INFORMATION AGENCY
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA

UNCLASSIFIED

TECHNICAL INFORMATION SERIES

R63SD4

MAGNETOHYDRODYNAMIC BOUNDARY LAYERS *

A. SHERMAN
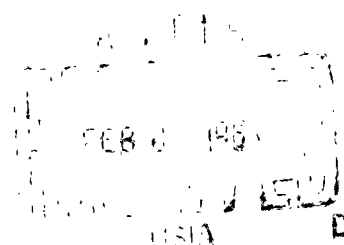
SPACE SCIENCES LABORATORY

GENERAL ELECTRIC

MISSILE AND SPACE DIVISION

SPACE SCIENCES LABORATORY

AEROPHYSICS SECTION

MAGNETOHYDRODYNAMIC BOUNDARY LAYERS[*]

by

A. Sherman

R63SD4
January, 1963

MISSILE AND SPACE DIVISION

GENERAL ⚫ ELECTRIC

CONTENTS

FOREWORD

This report has been written as Chapter 12

of a forthcoming book, Engineering Magnetohydro-

dynamics. References to other chapters refer to

other chapters in that book.

12.1 INTRODUCTION

The subject matter to be covered in the present chapter will consist of Magnetohydrodynamic flows of a boundary layer character. In conventional fluid dynamics, boundary layer theory has been developed to a high degree of sophistication. No comparable extensive treatment will be attempted here. The problems which will be discussed will be chosen to illustrate those new features introduced by the interaction between currents flowing in the conducting fluid or plasma and the electromagnetic field. The principle interest in the present chapter will be in flows for which the magnetic Reynolds number is vanishingly small. For completeness, however, the boundary layer character of flows with $R_m \to \infty$ will be treated in the last section.

The first problem considered will be the Rayleigh problem in which a magnetic field is applied normal to the surface of an impulsively moved half plane. Its principle value will be in that a solution can be obtained in closed form so that the nature of Magnetohydrodynamic boundary layer flows can be inferred. Following this introductory treatment the boundary layer equations are obtained from the full equations by means of the well known boundary layer approximation. Based on these equations, incompressible boundary layer flows are examined first. Two methods of solution of the non-linear partial differential equations are developed. For flows which satisfy certain specified conditions, similarity exists and the equations can be reduced to an ordinary differential equation which must then be integrated numerically. For more realistic boundary conditions similarity cannot exist so that in this instance series solutions are necessary. For high speed flows (hypersonic) compressibility effects must be considered, so compressible boundary layers are considered next. In this case the boundary conditions are such that similarity conditions can be applied and the solution to the problem reduces to that of numerical integration of two coupled ordinary differential equations.

To conclude this chapter the electric current and induced magnetic field boundary layers which can form on surfaces when the magnetic Reynolds number is very high are considered. Although the Magnetic Reynolds is extremely small in almost all cases of interest in continuum Magnetohydrodynamics the above subject is of interest since it involves application of old boundary layer concepts to a new physical problem.

12.2 THE RAYLEIGH PROBLEM

The problem which will be studied in the present section is shown in figure 12-1.
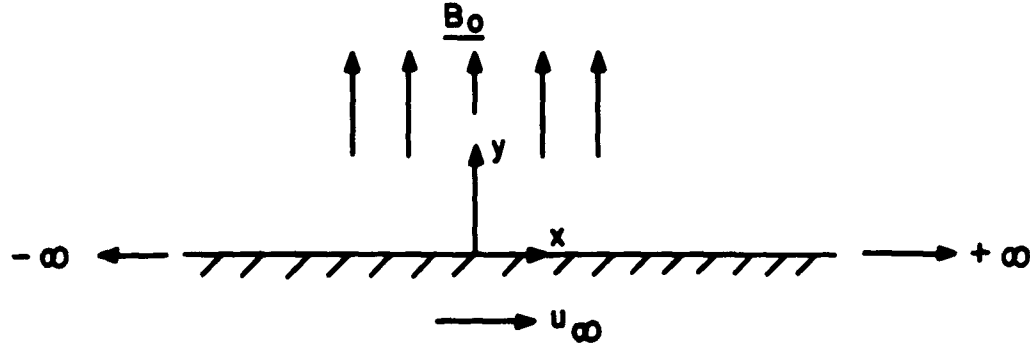

Figure 12-1. Rayleigh Problem Configuration

A field $\underline{B}_o$ is applied perpendicular to the plate and is assumed to be uniform in space and constant in time. The doubly infinite plate (a non-conductor) is assumed to move in x direction impulsively at $t = 0$. The problem is to determine the induced flow and magnetic field due to the impulsive motion in general.

A number of authors have investigated the general problem[1],[2] and have calculated the coupled flow and induced magnetic fields by means of various approximations. Of principle interest in the present chapter will be flows in which $R_m \ll R_e$. In this case the problem reduces to one in which the flow disturbance is confined to a thin boundary layer region while the induced field varies over a much larger region.

The equations appropriate to the present problem are essentially the same as those for the transient couette flow, except that now the second, stationary, wall is infinitely far away:

$$\rho \ \frac{\partial u}{\partial t} = \frac{B_o}{\mu} \ \frac{\partial b_x}{\partial y} + \eta \ \frac{\partial^2 u}{\partial y^2} \qquad (12.1)$$

$$\frac{\partial b_x}{\partial t} = B_o \ \frac{\partial u}{\partial y} + \frac{1}{\sigma\mu} \ \frac{\partial^2 b_x}{\partial y^2} \qquad (12.2)$$

The boundary and initial conditions are

$$t = 0: \quad u(y, 0) = 0 \qquad b_x(y, 0) = 0$$

$$t > 0: \quad u(0, t) = u_o \qquad b_x(0, t) = 0$$

where, of course, u and b_x must remain finite as $y \to \infty$. The requirement that $b_x = 0$ on the moving non-conductive wall is arrived at in much the same way as was done for Couette flow, see Chapter 10 section 10.3.

The above equations can be most readily treated with the aid of the Laplace transform defined by

$$L \left\{ f(y, t) \right\} = \hat{f}(y, s) = \int_o^\infty e^{-st} f(y, t) \, dt$$

where s is some constant parameter. Taking the Laplace transform of (12.1) and (12.2) gives

$$s\hat{u} = \frac{B_o}{\rho \mu} \frac{d\hat{b}_x}{dy} + \nu \frac{d^2 \hat{u}}{dy^2} \tag{12.3}$$

$$s\hat{b}_x = B_o \frac{d\hat{u}}{dy} + \frac{1}{\mu \sigma} \frac{d^2 \hat{b}_x}{dy^2} \tag{12.4}$$

where $\nu = \eta/\rho$ = kinematic viscosity and the boundary conditions are now

$$\hat{u}(o, s) = \frac{u_o}{s} \qquad \hat{b}_x(o, s) = 0$$

The solution of equations (12.3) and (12.4) is readily found to be

$$\hat{u} = \frac{(\frac{m^2}{\sigma \mu} - s)}{m B_o} A \, e^{-my} + \frac{(\frac{n^2}{\sigma \mu} - s)}{n B_o} B \, e^{-ny} \tag{12.5}$$

$$\hat{b}_x = A \, e^{-my} + B \, e^{-ny} \tag{12.6}$$

where A and B are arbitrary constants. Terms which diverge at infinity have been omitted, and $\pm m$ and $\pm n$ are solutions of

$$(\frac{r^2}{\mu \sigma} - s)(\nu r^2 - s) - \frac{B_o^2}{\rho \mu} r^2 = 0 \tag{12.7}$$

which is satisfied if

$$m = \sqrt{\frac{\sigma B_o^2}{4\eta} + \alpha s} + \sqrt{\frac{\sigma B_o^2}{4\eta} + \beta s}$$

$$n = \sqrt{\frac{\sigma B_o^2}{4\eta} + \alpha s} - \sqrt{\frac{\sigma B_o^2}{4\eta} + \beta s}$$

where

$$\alpha = \frac{\left(\dfrac{1}{\sqrt{\mu\sigma}} + \sqrt{\nu}\,\right)^2}{4\,\dfrac{\nu}{\mu\sigma}} \qquad\qquad \beta = \frac{\left(\dfrac{1}{\sqrt{\mu\sigma}} - \sqrt{\nu}\,\right)^2}{4\,\dfrac{\nu}{\mu\sigma}}$$

The constants A and B must be selected to satisfy the boundary conditions imposed. The solution then assumes the following form:

$$\frac{\hat{u}}{u_\infty} = \frac{1}{s\,(m-n)\left[\dfrac{1}{\mu\sigma} + \dfrac{s}{mn}\right]} \left\{ \frac{(\dfrac{m^2}{\mu\sigma} - s)}{m}\,e^{-my} - \frac{(\dfrac{n^2}{\mu\sigma} - s)}{n}\,e^{-ny} \right\} \tag{12.8}$$

$$\frac{\hat{b}_x}{B_o} = \frac{u_\infty}{s\,(m-n)\left[\dfrac{1}{\mu\sigma} + \dfrac{s}{mn}\right]} \left\{ e^{-my} - e^{-ny} \right\} \tag{12.9}$$

Inversion of these equations is quite difficult as they stand. Since our principle interest will be in flows in which $R_m \to 0$, and this implies

$$P_m = \sqrt{\frac{R_m}{R_e}} \to 0 \quad \text{or} \quad \frac{1}{\sigma\mu} \gg \nu \;,$$ a solution of boundary layer character can be anticipated and the problem accordingly simplified. Keeping $\sigma\mu$ finite, and letting $\nu \to 0$ while holding $y/\sqrt{\nu}$ fixed leads to the following results:

$$m \to \left[\frac{\sigma B_o^2}{\eta} + \frac{s}{\nu}\right]^{1/2} \qquad\qquad n \to s\left[\frac{s}{\sigma\mu} + \frac{B_o^2}{\rho\mu}\right]^{-1/2}$$

and

$$\frac{\hat{u}}{u_\infty} = \frac{1}{s}\,e^{-\sqrt{\dfrac{\sigma B_o^2}{\eta} + \dfrac{s}{\nu}}\;y} \tag{12.10}$$

$$\hat{b}_x = 0$$

Accordingly, when $b_x = 0$ at the moving wall it is zero throughout the velocity boundary layer region. Such a result could have been anticipated. If on, the other hand, other boundary conditions had been selected for b_x other results would have been obtained. If the problem geometry were such that for long times b_x (0, t) $\cong$ constant rather than zero then the boundary condition would be that b_x at the boundary be some function of time depending on the total current flow. Such a boundary condition could not be conveniently handled. Finally, if the moving wall is assumed to be a perfect conductor the electric field in the fluid adjacent to the wall must be zero. Then from equations (10.1) and (10.5a)

we have

$$- \frac{\partial b_x}{\partial y} = \sigma \left[E_z + u B_o \right]$$

or

$$\frac{\partial b_x}{\partial y} (o, t) = - \sigma u_\infty B_o \qquad (12.11)$$

which is the required boundary condition for this case. As has been shown by Ludford[1] the solution when equation (12.11) is used as the boundary condition leads to a b_x which is constant throughout the boundary layer region.

Returning now to the problem originally posed, the inverse of equation (12.10) is readily found to be

$$\frac{u}{u_\infty} = \frac{1}{2} \left[e^{-y\sqrt{\frac{\sigma B_o^2}{\mu}}} \, \text{erfc} \left(\frac{y}{2\sqrt{\nu t}} - \sqrt{\frac{\sigma B_o^2}{\rho} t} \right) \right.$$

$$\left. + e^{y\sqrt{\frac{\sigma B_o^2}{\mu}}} \, \text{erfc} \left(\frac{y}{2\sqrt{\nu t}} + \sqrt{\frac{\sigma B_o^2 t}{\rho}} \right) \right] \qquad (12.12)$$

This result is identical to that of Rossow[2] although the induced magnetic field was not considered in that reference. The reason for the agreement is obvious as $b_x = 0$ throughout the boundary layer region. However, it should be noted that b_x will become finite above the boundary layer and approach some constant value at ∞. The value of formulating the complete problem before making the boundary layer approximation will be seen shortly when the shear stress at the wall is calculated.

First, however, consider the solution for u found above. If B_o is allowed to go to zero equation (12.12) reduces to the classical Rayleigh problem:

$$\frac{u}{u_\infty} = 1 - \text{erf} \left(\frac{y}{2\sqrt{\nu t}} \right) \qquad (12.13)$$

In this case the velocity profile depends on only one variable $\left(\frac{y}{2\sqrt{\nu t}} \right)$ which is essentially the similarity variable to be discussed later. In the presence of magnetohydrodynamic effects such similitude no longer exists and there is no one variable on which $\frac{u}{u_\infty}$ depends. In order to illustrate the flow pattern, calculations based on equation (12.12) have been carried out and are presented in Figure 12-2.
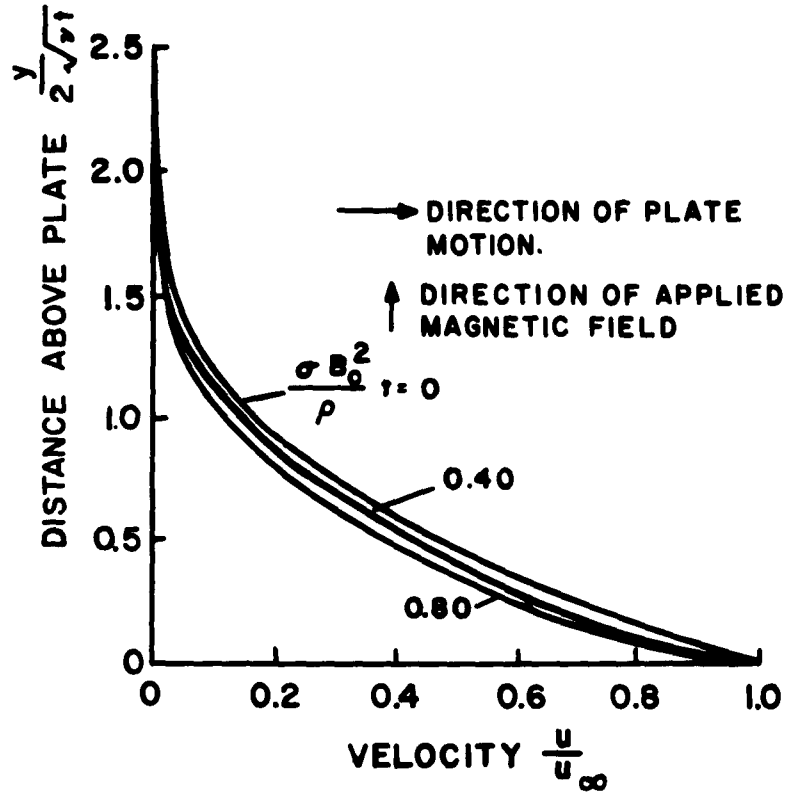
Figure 12-2. Velocity Profiles in the Rayleigh Problem

It can be seen that application of the magnetic field increases the time required for the flow velocity above the plate to reach any specific value.

If desired, the above solution can be interpreted as a boundary layer flow with $t \rightarrow x/u_\infty$. In this case we observe that as the field strength is increased the velocity profile becomes fuller. This result is analagous to that found for the Hartmann flow earlier. Also, the above results suggest that a natural variable to formulate the boundary layer problem with should be $\eta = \dfrac{y}{\sqrt{\nu x}}$ just as in the in the case of conventional incompressible boundary layers.

Finally, the force on the moving plate will be considered. The shear stress is given simply as

$$(\tau)_\omega = \eta \left(\frac{\partial u}{\partial y} \right)_{y=0} \tag{12.14}$$

Now from equation (12.8) the transform of τ_ω may be derived

$$\hat{\tau}_\omega = \left(\frac{\eta\, u_\infty}{1 + \sqrt{P_m}} \right) \frac{\sqrt{\dfrac{\sigma B_0^2}{\eta} + \dfrac{s}{\nu}\,(1+\sqrt{P_m})^2}}{s} \tag{12.15}$$

The inverse of this expression gives the following result

$$\tau_\omega (t) = \frac{\eta u_\infty}{\sqrt{\nu}} \left\{ \frac{1}{\sqrt{\pi t}} e^{-\gamma t} + \sqrt{\gamma} \, \text{erf} \sqrt{\gamma t} \right\} \tag{12.16}$$

where

$$\gamma = \frac{\sigma B_o^2}{\rho (1 + \sqrt{P_m})^2}$$

which agrees with Rossows[2] result when $P_m = 0$. The above equation is valid for arbitrary values of P_m and is thus a much more general result.

In order to illustrate the influence of P_m on the solution, equation (12.16) is plotted in figure 12-3.

A number of valuable observations can be made based on these results. First, it is clearly seen that as σ increases the wall shear becomes larger for any given time. This increased shear arises as a result of the enhanced Lorentz force on the conducting fluid near the moving wall. At very long times all of the curves of Figure 12-2 reach asymptotic values given by

$$\lim t \to \infty \quad (\tau_\omega') = \frac{\sqrt{P_m}}{1 + \sqrt{P_m}}$$

Now in the non-magnetohydrodynamic Rayliegh problem equation (12.13) shows that $u \to \infty$ as $t \to \infty$ so that $\tau_\omega \to 0$, and it is of interest to inquire as to why in the present problem $\tau_\omega' \neq 0$ in the same limit. If we return to the original equations of the problem, equations (12.1) and (12.2), and assume $\frac{\partial}{\partial t} = 0$ the following solution is obtained for u and b_x.

$$\frac{u}{u_\infty} = e^{-\sqrt{\frac{\sigma B_o^2}{\eta}} \, y}$$

$$b_x = \sqrt{\sigma \eta} \, u_\infty \mu \left[\frac{u}{u_\infty} - 1 \right]$$

The steady state long time solution then has an exponential velocity profile rather than the constant one when say $\sigma = 0$. This shape of the profile arises because of the existence of the currents in the fluid and the attendent Lorentz force. The form of the current distribution is also exponential decreasing from some finite value at the wall to zero at infinity.
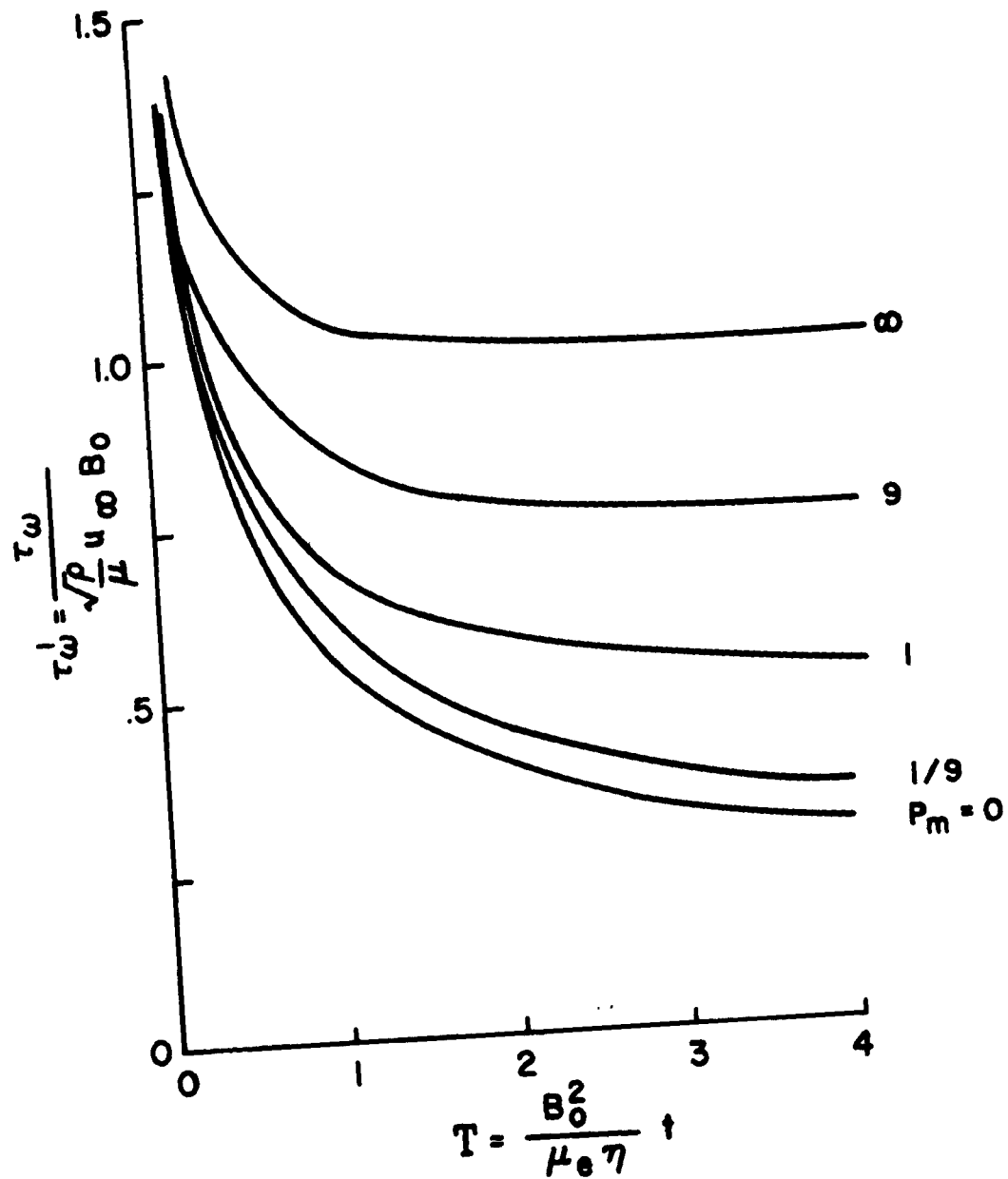
Figure 12-3. Wall Shear in Rayliegh Problem

From the point of view of the physical problem the wall shear is balanced by the reaction force on the source of the magnetic field. When there is no magnetic field there is nothing to balance this shear and so it must be zero in the steady state.

12.3 FORMULATION OF BOUNDARY LAYER EQUATIONS

The boundary layer approximation in fluid mechanics has been discussed by many authors from the time of Prandtl; there is, therefore, no justification in repeating these arguments. What will be discussed in the present section will be those new terms which appear in the equations due to Magnetohydrodynamic effects, and their simplification via the boundary layer approximation. Attention for the present will be restricted to flows with $R_m \sim 0$ so that the magnetic field can be taken as the applied field (i.e. section 10.8 of chapter 10). The question of the boundary layer approximation when $R_m \to \infty$ will be discussed later.

The general problem which will be considered is shown in Figure 12-4. The magnetic field vector is assumed to lie in the xy plane but may have x and y components. The Lorentz force for such a two dimensional problem is given by

$$\underline{F} = \underline{j} \times \underline{B}_0 = \sigma[\underline{E} + \underline{q} \times \underline{B}] \times \underline{B} \qquad (12.17)$$
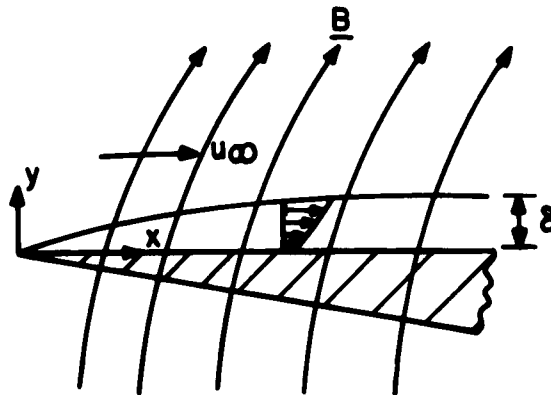
Figure 12-4. Magnetohydrodynamic Boundary Layer Problem

where the current is derived from the Ohm's law since induced magnetic fields have been assumed negligible. Now since only steady flow problems will be considered it will be valid to take $\underline{E} = 0$ or if desired some constant. It may be recalled that in the unsteady problem (i.e. transient Couette flow) it is not in general valid to assume $\underline{E} = 0$. For the present case, equation (12.17) can be written as

$$F_x = -\sigma (u B_y^2 - v B_x B_y) \tag{12.18}$$

$$\dot{F}_y = \sigma (u B_x B_y - v B_x^2) \tag{12.19}$$

For purposes of order of magnitude arguments one can assume that $B_y = \mathcal{O}(B_x)$. Then, since $\frac{v}{u} = \mathcal{O}(\delta)$ where δ is the boundary layer thickness the first of the above expressions can be written as

$$F_x \cong -\sigma B_y^2 u \tag{12.20}$$

where B_y can be a function of x and y in general.

In a boundary layer calculation it will be exceedingly inconvenient to keep B_y a function of y. Accordingly, it is of interest to study B_y (x, y) somewhat more carefully. For a two dimensional field one of Maxwell's equations ($\underline{\nabla} \cdot \underline{B} = 0$) requires

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

Now with $B_x = \mathcal{O}(B_y)$ and $\frac{\partial}{\partial x} = \mathcal{O}(1)$ while $\frac{\partial}{\partial y} = \mathcal{O}(\delta^{-1})$ we have

$$\frac{\partial B_y}{\partial y} = \mathcal{O}(1)$$

or

$$\Delta B_y = \mathcal{O}(\delta)$$

Thus, the change in B_y across the boundary layer will be the order of δ and can be neglected.

Making use of equation (12.20), assuming $p = p(x)$ and making the usual boundary layer approximations the x component of the momentum equation becomes: becomes:

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right) + \frac{dp}{dx} = -\sigma B_y^2 u + \frac{\partial}{\partial y}\left(\eta \frac{\partial u}{\partial y}\right) \tag{12.21}$$

Before proceeding further the assumption that $p = p(x)$ must be examined further. Again assuming $B_y = \mathcal{O}(B_x)$ equation (12.19) reduces to

$$F_y \cong \sigma u B_y B_x \qquad (12.22)$$

Now the y component of the momentum equation can be written in the following form

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = Q u B_x B_y + \frac{1}{R_e} \frac{\partial}{\partial y} \left(\eta \frac{\partial v}{\partial y} \right) \qquad (12.23)$$

where

$$Q = \frac{\sigma B_R^2 L}{\rho_R u_\infty}$$

$$R_e = \frac{\rho_R u_\infty L}{\eta_R}$$

and the velocities have been made dimensionless by some reference velocity (u_∞), the distances x and y by some reference length (L), the pressure by $\rho_0 u_\infty^2$, the density ρ and viscosity η by reference values, and the magnetic field by some reference field (B_R). For magnetohydrodynamic boundary layer problems of interest $Q = \mathcal{O}(1)$. Then an order of magnitude analysis of equation (12.23) with $u = \mathcal{O}(1)$, $v = \mathcal{O}(\delta)$, $\frac{\partial}{\partial x} = \mathcal{O}(1)$, $\frac{\partial}{\partial y} = \mathcal{O}(\frac{1}{\delta})$, and $R_e = \mathcal{O}(\frac{1}{\delta^2})$ gives

$$\frac{\partial p}{\partial y} = \mathcal{O}(1)$$

or

$$\Delta p = \mathcal{O}(\delta) \sim 0$$

Thus, the pressure change across the boundary layer is the order of δ (a small quantity) and can be neglected. It should be noted, however, that in the absence of magnetic forces $\Delta p = \mathcal{O}(\delta^2)$ so that it is less valid to assume $p = p(x)$ in the present case than it had been in the absence of a magnetic field.

Next, the Joule heating term which will enter into the energy equation must be considered.

$$H = \frac{i^2}{\sigma} = \sigma (\underline{q} \times \underline{B}) \cdot (\underline{q} \times \underline{B}) \qquad (12.24)$$

or

$$H = \sigma \left[u^2 B_y^2 + v^2 B_x^2 - 2uv B_x B_y \right]$$

as before, the last two terms can be neglected since $v/u = \mathcal{O}(\delta)$. Thus,

$$H \cong \sigma u^2 B_y^2$$

and the energy equation can be written as follows

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} - u \frac{dp}{dx} =$$

$$\frac{\partial}{\partial y} \left(\frac{\eta}{P_R} \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial y} \left(\frac{5 k j h}{2 e C_P} \right)$$

$$+ \sigma u^2 B_y^2 + \eta \left(\frac{\partial u}{\partial y} \right)^2 \tag{12.25}$$

where

$$P_R = \frac{C_P \eta}{k}$$

Finally, the mass continuity equation must be added to complete the system
of equations:

$$\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 \tag{12.26}$$

Before proceeding to the solution of specific problems it will be of interest
to study the general question of similar solutions for incompressible, constant
property, boundary layer flows. Consider equation (12.21) evaluated in the inviscid
free stream flow

$$\rho u_\infty \frac{d u_\infty}{dx} + \frac{dp}{dx} = - \sigma B_y^2 u_\infty \tag{12.27}$$

Combining this with equation (12.21) leads to the following relation.

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - u_\infty \frac{du_\infty}{dx} + \frac{\sigma B_y^2}{\rho} (u - u_\infty) = \nu \frac{\partial^2 u}{\partial y^2} \tag{12.28}$$

Introducing $u = \frac{\partial \psi}{\partial y}$, and $v = - \frac{\partial \psi}{\partial x}$ satisfies continuity and equation (12.28)
becomes

$$\psi_y \psi_{yx} - \psi_x \psi_{yy} + \frac{\sigma B_y^2}{\rho} (\psi_y - u_\infty) = u_\infty \frac{d u_\infty}{dx} + \nu \psi_{yyy} \tag{12.29}$$

The problem now is to determine whether or not the above can be reduced to an
ordinary differential equation by a proper choice of variables. Let us assume

$$\psi (x, y) = \sqrt{u_\infty \nu x} \ \ f (\eta)$$

where

$$\eta (x, y) = \frac{y}{2} \sqrt{\frac{u_\infty}{\nu x}}$$

and where it should be noted that u_∞ and B_y are known functions of x as yet
unspecified. Substituting the above equation (12.29) becomes

$$2 \frac{du_\infty}{dx} (f')^2 - \left(\frac{u_\infty}{x} + \frac{du_\infty}{dx} \right) ff''$$

$$- \frac{8 \sigma B^2}{\rho} \left(1 - \frac{1}{2} f' \right) = \frac{8 du_\infty}{dx} + \frac{u_\infty}{x} f''' \tag{12.30}$$

Accordingly, similarity can be achieved if

$$\frac{du_\infty}{dx} \ \alpha \ \frac{u_\infty}{x}$$

and

$$B_y \ \alpha \ \sqrt{\frac{u_\infty}{x}}.$$

so that u_∞ and B_y must be of the following form to permit a similar solution

$$u_\infty = C_1 x^m$$

$$B_y = C_2 x^{\frac{m-1}{2}}$$

In fluid mechanics the x^m variation of u_∞ corresponds to the wedge flow solutions. In magnetohydrodynamic problems such a simple interpretation is not possible since the applied magnetic field interacts with the inviscid free stream flow. We will return to this point in the next section.

A final point in regard to the above treatment should be made. In looking for similar solutions only the incompressible boundary layer was studied. For a compressible flow the situation is much more complex. Now, for a similar solution to be found for the compressible boundary layer problem it is safe to say that a minimum condition would be that u_∞ and B_y satisfy the conditions already established for the incompressible case. In addition, many other assumptions and requirements will be necessary to obtain similarity in the compressible case. Some of the new phenomena associated with compressible magnetohydrodynamic boundary layer flows will be treated in section 12.5. The following section will be restricted to incompressible boundary layers.

12.4 INCOMPRESSIBLE BOUNDARY LAYERS

The distinguishing feature of an incompressible magnetohydrodynamic boundary layer is that the inviscid flow external to the boundary layer is also a conductor. Then, since it is not possible to restrict the magnetic field to the boundary layer region alone, one has no choice but to incorporate the results of the inviscid magnetohydrodynamic analysis into the boundary layer

investigation. On the other hand, this may not be necessary in compressible flows in which the conductivity can vary, and for certain types of flows may be assumed zero external to the boundary layer. Application of the similarity solution to incompressible boundary layer although feasible, in principle, will not be carried out due to the difficulty of interpreting the resulting inviscid flow. Instead, two simple incompressible boundary layer problems will be analyzed by the series expansion technique

Boundary Layer with Uniform Free Stream

The first problem that will be considered will take as a model for the inviscid flow the flow through a parallel walled two dimensional channel and will consider the flat plate situated somewhere in the channel some distance from either wall. In this case, the free stream velocity will be a constant and the pressure gradient will be given by $dp/dx = -\sigma u_\infty B_y^2$. The applied magnetic field will be assumed constant so that the pressure will vary linearly along the channel. The equation to be solved is equation (12.29) which simplifies to the following.

$$\psi_y \psi_{yx} - \psi_x \psi_{yy} + \frac{\sigma B_y^2}{\rho} (\psi_y - u_\infty) = \nu \psi_{yyy} \tag{12.31}$$

where the boundary conditions are

$$\psi = \psi_y = 0 \qquad \text{at} \quad y = 0$$

$$\psi_y = u_\infty \qquad \text{at} \quad y = \infty, \ x = 0$$

Since the pressure gradient in the present problem is constant and favorable it will be adequate to use a classical Blasius series expansion to effect a solution. Assume

$$\psi(x,y) = \sqrt{u_\infty \nu x} \left[f_0 + \frac{\sigma B_y^2}{\rho} x f_1 + \left(\frac{\sigma B_y^2}{\rho} x \right)^2 f_2 + \dots \right] \tag{12.32}$$

where it is assumed that $\left(\frac{\sigma B_y^2}{\rho} x \right)$ is a small quantity and the f's are functions

of $\eta = y\sqrt{\frac{u_\infty}{\nu x}}$. Introduction of the above assumption for ψ into equation (12.31) leads to the following infinite set of ordinary differential equations.

$$f_0''' + f_0 f_0'' = 0, \tag{12.33}$$

with the following boundary conditions:

$$f_0(o) = f_0'(o) = 0 \qquad f_0'(\infty) = 1$$

$$f_1''' + \frac{1}{2} f_o' f_1'' - f_o' f_1' + \frac{3}{2} f_o'' f_1 = f_o' - 1 \tag{12.34}$$

with

$$f_1 (o) = f_1' (o) = 0 \qquad f_1' (\infty) = 0$$

$$f_2''' + \frac{1}{2} f_o f_2'' - 2f_o' f_2' + \frac{5}{2} f_o'' f_2 = (f_1')^2 - \frac{3}{2} f_1 f_1'' + f_1'$$
$$\tag{12.35}$$

with

$$f_2 (o) = f_2' (o) = 0 \qquad f_2' (\infty) = 0$$

etc.

The first of these is the well known Blasius equation for which the solution has been tabulated for the boundary conditions shown. All subsequent equations are linear, but since they depend on the preceeding solutions numerical integration is necessary. The first two equations beyond the Blasius have been solved by Rossow[2]. The resulting velocity profiles are shown in figure 12-5.
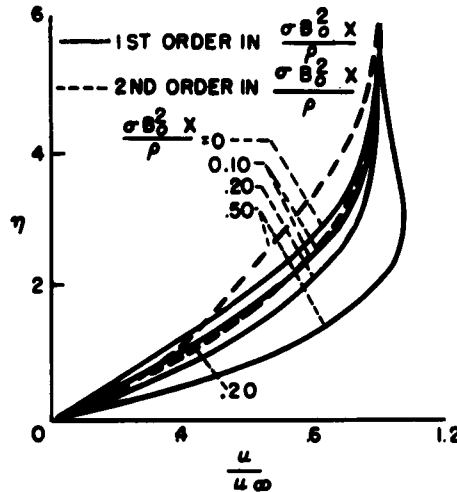

Figure 12-5. Incompressible Boundary Layer for Uniform Applied Magnetic Field and Free Stream Velocity

Again these results are in agreement with the Hartmann flow treated in Chapter 9 and the Rayliegh problem solution discussed earlier in the present chapter. In other words, the presence of the applied magnetic field tends to make the velocity profile fuller. If, on the other hand, the free stream velocity had been decreasing with x the pressure gradient would have become unfavorable, the boundary layer profile would have become less full, and the possibility of flow separation would exist. Such a case will be considered next.

Boundary Layer Subject to Adverse Pressure Gradient

Consider, as an appropriate inviscid flow the flow through a parallel walled channel when the applied magnetic field is created by a current-carrying wire alligned perpendicular to the flow direction and imbedded in the lower wall. As shown in Figure 12-6 the boundary layer development along the lower wall will be considered.
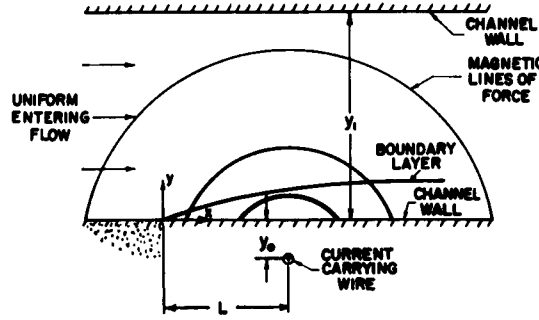

Figure 12-6. Boundary Layer Development with Magnetic Field
Generated by Current-Carrying Wire in Lower Wall

One must, of course, assume that the boundary layer ahead of $x = 0$ can be removed, perhaps by a bleed port shown schematically as the dotted region. The equation and boundary conditions for ψ have already been presented, equation (12.31), and will also be used in the present problem. One should be aware, however, that requiring $\psi_y = u_\infty$ at $y = \infty$ means that the inviscid flow shear (see Figure 10-33) at the lower wall is being neglected. In general, this shear can be neglected when the inviscid free stream flow has been linearized.

As noted earlier, the existence of a non-uniform free stream velocity, and the possibility of separation, requires the use of a series solution which is more sophisticated. A procedure frequently used for conventional boundary layers and ideally suited to the present problem is due to Görtler. Its extension to magnetohydrodynamics and its application to the present problem will be carried out now.

If the following dimensionless variables are defined

$$\tilde{x} = \frac{xQ}{y_o} \qquad \tilde{y} = \frac{y}{y_o}\sqrt{R_e Q} \qquad \tilde{u}_\infty = \frac{u_\infty}{u_i}$$

$$\tilde{\psi} = \frac{\psi\sqrt{R_e Q}}{y_o u_i} \qquad \tilde{B}_y = \frac{B_y}{B_o}$$

and
$$R_e = u_i \, y_0 \, \nu \qquad\qquad Q = \frac{\sigma B_0^2 \, y_0}{\rho \, u_i}$$

where $u_i = u(o,y)$, assumed constant, y_0 is some reference length, and B_0 some reference magnetic field, and the following new independent and dependent variables are defined

$$\xi = \int_0^{\tilde{x}} \tilde{u}_\infty \, d\tilde{x} \qquad\qquad \eta = \frac{\tilde{u}_\infty \, \tilde{y}}{\sqrt{2\xi}}$$

$$F = \tilde{\psi} \big/ \sqrt{2\xi}$$

Then equation (12.31) now becomes

$$F_{\eta\eta\eta} + F \, F_{\eta\eta} + \left[\frac{2\xi \frac{d\tilde{u}_\infty}{dx}}{\tilde{u}_\infty^2} \right] (1 - F_\eta^2)$$

$$= 2\xi \left[F_\eta \, F_{\xi\eta} - F_\xi \, F_{\eta\eta} + \frac{\tilde{B}_y^2}{\tilde{u}_\infty^2} (F_\eta - 1) \right] \tag{12.36}$$

with the following boundary conditions

$$F = F_\eta = 0 \qquad\qquad \text{at } \eta = 0$$

$$F_\eta = 1 \qquad\qquad \text{at } \eta = \infty$$

The problem can be solved in general by assuming the following expansions:

$$F(\xi,\eta) = F_0(\eta) + \xi \, F_1(\eta) + \xi^2 \, F_2(\eta) + \ldots$$

$$\frac{2}{\tilde{u}_\infty^2} \frac{d\tilde{u}_\infty}{d\tilde{x}} = \beta(\xi) = \beta_0 + \xi\beta_1 + \xi^2\beta_2 + \ldots.$$

$$\frac{\tilde{B}_y^2}{\tilde{u}_\infty^2} = g(\xi) = g_0 + \xi g_1 + \xi^2 g_2 + \ldots.$$

Substituting these into equation (12.36) one obtains, as before, an infinite set of equations. The first of these is the Blasius equation

$$F_0''' + F_0 \, F_0'' = 0 \tag{12.37}$$

$$F_0(o) = F_0'(o) = 0 \qquad F_0'(\infty) = 1$$

The second and all later equations are linear, and are given by the following recursive system

$$F_k''' + F_o F_k'' - 2k F_o' F_k' + (2k + 1) F_o'' F_k$$

$$+ R_{k-1} = 0 \qquad k = 1, 2, 3, \ldots ,$$

(12.38)

where

$$R_{k-1} = \beta_{k-1} (1 - F_o'^2) - \sum_{j=1}^{k-1} \sum_{i=1}^{k-j} \beta_{j-1} F_i' F_{k-i-j}'$$

$$- F_o' \sum_{j=1}^{k-1} \beta_{j-1} F_{k-j}' - 2 \sum_{j=1}^{k-1} (k - j) F_j' F_{k-j}'$$

$$+ \sum_{i=1}^{k-1} (1 + 2j) F_j F_{k-j}'' + 2g_{k-1} (1 - F_o')$$

$$- 2 \sum_{j=1}^{k-1} g_{j-1} F_{k-j}'$$

and

$$F_k = F_k' = 0 \text{ at } \eta = 0$$

$$F_k' = 0 \qquad \text{at } \eta = \infty.$$

Application of the above equations to the solution of the specific problem of Figure 12-6 has been carried out by Sherman[3]. Numerical results were obtained for an inviscid flow in which $L/y_o = 1.6$ and $Q = 0.1$. Curves showing the wall skin friction, boundary layer thickness, and velocity profiles are shown in Figures 12-7 and 12-8.
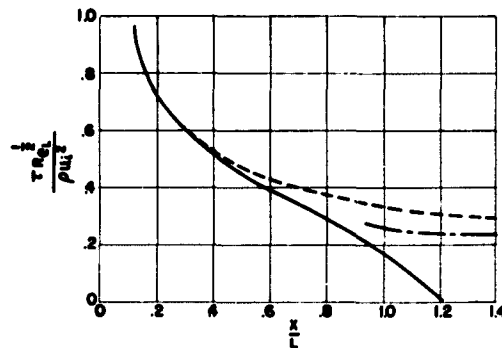

Figure 12-7. Wall Skin Friction Versus Distance From Leading Edge for $Q = 0.1$ and $L/y_0 = 1.6$. Nonmagnetic case ----; Magnetic Case, Series Solution ——; Magnetic Case, Integral Approximation ——·——.
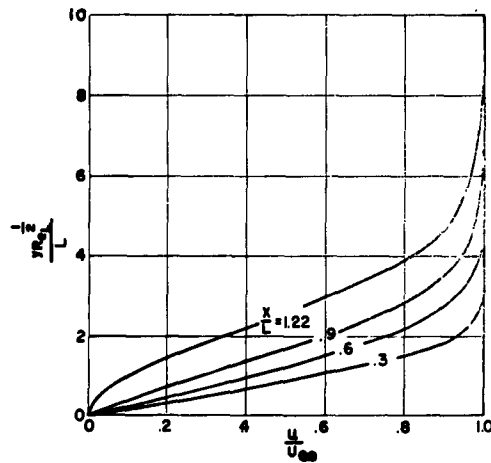
Figure 12-8. Boundary Layer Velocity Profiles at Several Positions
Downstream of Leading Edge for Q = 0.1 and L/y_0 = 1.6

In order to interpret these results properly it will be of value to review
the physical phenomena occuring in the flow. First, the applied magnetic field
acts on the inviscid flow in such a way as to cause the boundary layer free stream
velocity to decrease rapidly in the vicinity of the wire. This tends to make the
velocity profile less full, and thereby reduce the wall skin friction. In addition,
the magnetic field within the boundary layer creates a Lorentz force which tends
to retard the flow. This effect then also tends to reduce the skin friction, and
both combined may retard the boundary layer sufficiently to cause separation.
That such flow separation is indeed a practical possibility is shown in Figures
12-7 where τ = 0 at x = 1.22 L, just beyond the wire. For the present strongly
retarded flow the classical Karman-Pohlhausen integral approach is seen to be a
poor approximation. The details of the velocity profiles are shown in Figure 12-8
where it must be kept in mind that u_∞ is decreasing as x increases from zero.

12.5 COMPRESSIBLE BOUNDARY LAYERS

Although some of the requirements for similar magnetohydrodynamic boundary
layers had been identified earlier such solutions were not sought in the incom-
pressible case due to the difficulty in interpreting the results in terms of a
practical problem. When the flow is compressible, however, such difficulties
may not exist. For example, when considering the hypersonic flow over a flat
plate in which the free stream is at a low temperature compared to the high
temperatures in the boundary layer, one may assume σ_∞ = 0 so that the applied

magnetic field does not disturb the inviscid flow and u_∞ = constant. Also, there are some practically important problems for which the interaction between the magnetic field and the inviscid flow have been calculated and have shown u_∞ to have the required form for similarity.

In the cases just cited similar solutions have practical significance, and in fact are absolutely essential for progress to be made in solving these complex problems. In the present section two compressible boundary layer flows will be treated in some detail, and a third will be discussed.

First, consider the hypersonic flow over a semi-infinite flat plate when the temperature within the boundary layer is high enough to ionize the gas. In order to obtain a similar solution, the free stream velocity is assumed constant (low free stream temperature so that $\sigma = 0$), and the applied magnetic field B_y is assumed to vary inversely as the square root of x. In addition, the wall temperature is assumed constant, and the gas is considered to be in thermodynamic equilibrium. The geometry and coordinates of Figure 12-4 will be used.

Neglecting heat flux due to diffusion of species (current does not flow across a temperature gradient), the Hall effect, and induced magnetic fields, and putting $\frac{dp}{dx} = 0$ the boundary layer equations are given by Equations (12.21), (12.25), and (12.26). The magnetic field can be expressed as

$$B_y = \frac{B_0 \sqrt{L_0}}{\sqrt{x}}$$

The reduction of these equations to the ordinary differential equations corresponding to similarity is accomplished by the Crocco transformation[4]. The procedure will be sketched briefly.

If the independent variables of the problem are changed from x and y to x and u and $\tau \equiv \mu \frac{\partial u}{\partial y}$ the transformation formulas can be written as follows:

$$\left(\frac{\partial}{\partial x}\right)_y = \left(\frac{\partial}{\partial x}\right)_u + \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial}{\partial u}\right)_x$$

$$\left(\frac{\partial}{\partial y}\right)_x = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial}{\partial u}\right)_x = \frac{\tau}{\eta} \left(\frac{\partial}{\partial u}\right)_x$$

recognizing that we can consider y = y (x, u) the first of these leads to

$$0 = \left(\frac{\partial y}{\partial x}\right)_u + \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial y}{\partial u}\right)_x$$

or

$$\left(\frac{\partial u}{\partial x}\right)_y = - \frac{\left(\frac{\partial y}{\partial x}\right)_u}{\left(\frac{\partial y}{\partial u}\right)_x} = - \frac{y_x}{y_u}$$

and the second yields

$$1 = \frac{\tau}{\eta} \left(\frac{\partial y}{\partial u}\right)_x$$

or

$$y_u = \frac{\eta}{\tau}$$

Then the transformation equations can be rewritten to yield

$$\left(\frac{\partial}{\partial x}\right)_y = \left(\frac{\partial}{\partial x}\right)_u - \frac{y_x \tau}{\eta} \left(\frac{\partial}{\partial u}\right)_x$$

$$\left(\frac{\partial}{\partial y}\right)_x = \frac{\tau}{\eta} \left(\frac{\partial}{\partial u}\right)_x$$

Using these relations the transformed boundary layer equations become

$$\frac{\partial (\rho v)}{\partial u} = y_x \frac{\partial (\rho u)}{\partial u} - \frac{\eta}{\tau} \frac{\partial (\rho u)}{\partial x} \qquad (12.39)$$

$$\rho v = (\rho u) y_x + \frac{\partial \tau}{\partial u} - \frac{\sigma B_o^2 L_o \eta}{x} \frac{u}{\tau} \qquad (12.40)$$

$$\rho u \frac{\partial h}{\partial x} + \left(\frac{\tau}{\eta} \frac{\partial \tau}{\partial u} - \sigma B_o^2 L_o \frac{u}{x}\right) \frac{\partial h}{\partial u} =$$

$$+ \frac{\tau}{\eta} \frac{\partial}{\partial u} \left(\frac{\tau}{P_R} \frac{\partial h}{\partial u}\right) + \frac{\sigma u^2 B_o^2 L_o}{x} + \frac{\tau^2}{\eta} \qquad (12.41)$$

where the second term in the last of these has been simplified with the use of equation (12.40). Eliminating ρv from equation (12.40) by use of equation (12.39) gives

$$\rho u \left(\frac{\eta}{\tau}\right)_x + \frac{\eta}{\tau} (\rho u)_x + \frac{d^2 \tau}{du^2} - B_o^2 L_o \frac{d}{du} \left(\frac{\sigma \eta u}{\tau}\right) = 0$$

or

$$\left(\rho \frac{u \eta}{\tau}\right)_x + \frac{d^2 \tau}{du^2} - B_o^2 L_o \frac{d}{du} \left(\frac{\sigma \eta u}{\tau}\right) = 0 \qquad (12.42)$$

Next assume that

$$\tau (x, u) = \frac{G (u)}{\sqrt{x}}$$

and

$$h (x, u) = h (u)$$

so that equations (12.41) and (12.42) become

$$\frac{\rho u \eta}{2G} \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}} \frac{d^2 G}{du^2} - \frac{B_o^2 L_o}{\sqrt{x}} \frac{d}{du}\left(\frac{\sigma \eta u}{G}\right) = 0$$

or

$$G \frac{d^2 G}{du^2} = -\frac{\rho u \eta}{2} + G B_o^2 L_o \frac{d}{du}\left(\frac{\sigma \eta u}{G}\right) \tag{12.43}$$

and

$$\frac{G}{\eta x} \frac{dG}{du} \frac{dh}{du} - \frac{\sigma B_o^2 L_o u}{x} \frac{dh}{du} =$$

$$\frac{G}{\eta x} \frac{d}{du}\left(\frac{G}{P_R} \frac{dh}{du}\right) + \frac{\sigma u^2 B_o^2 L_o}{x} + \frac{G^2}{\eta x}$$

or

$$\frac{dG}{du} \frac{dh}{du} = \frac{d}{du}\left(\frac{G}{P_R} \frac{dh}{du}\right) + G$$

$$+ \frac{\sigma B_o^2 u L_o \eta}{G}\left[\frac{dh}{du} + u\right] \tag{12.44}$$

These are then two ordinary differential equations for G and h as functions of u. The numerical procedure for their solution is given in detail by Bush[5] where the method of expressing ρ, η, σ, and P_R as functions of h for high temperature air are also described. Some of the results of calculations for the constant wall temperature case are shown in Figure 12-9.
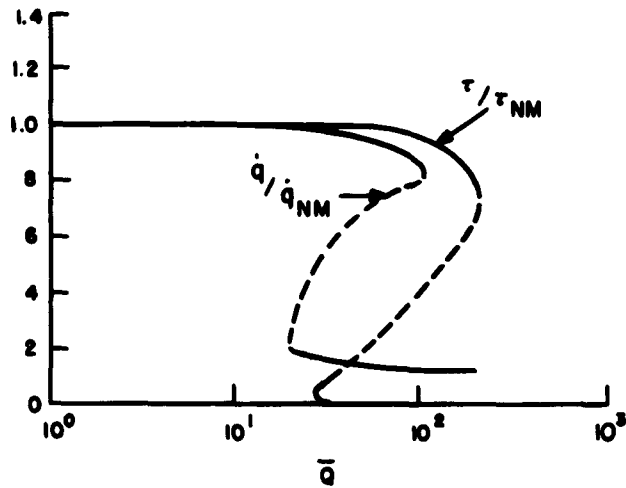

Figure 12-9. Shear Stress and Heat Flux for Hypersonic
Boundary Layer

The conditions chosen for the above case were $T_\infty = 222°K$, $p = 10^{-3}$ atm, $M_\infty = 25$, and a wall temperature of $\sim 2000°K$. The subscript NM stands for non-magnetic, and $\overline{Q}$ is the interaction parameter proportional to B_o^2.

The above results bear a striking resemblance to the real gas Couette flow results described in Chapter 10. Again, a hysterisis effect exists due to the form of the σ versus T curve and the dotted portion of the curves are unstable. The principle new information here is that in allowing the boundary layer to grow in height (in the couette flow δ is fixed) a reduction in heat flux of 80% is possible where little if any heat flux reduction had been predicted earlier. The shear stress does, however, behave as predicted.

Another compressible boundary layer flow of practical interest is one which grows along the electrode surface of a crossed field MHD channel. The principle new feature here is that a current flows normal to the boundary layer surface (see Figure 12-10) so that the contribution to the heat flux due to the diffusion of electrons in a temperature gradient cannot be neglected.
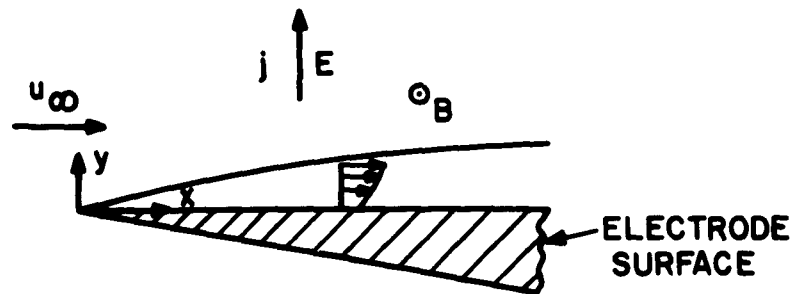
Figure 12-10.   Electrode Boundary Layer Configuration

Since the boundary layer under consideration will be developing within a finite width channel the usual assumption that the boundary layer thickness is not large enough to disturb the inviscid flow must be made. In addition, it is necessary to assume that the electrical resistance of the boundary layer is small compared to the resistance of the inviscid flow so that the overall current flow is determined external to the boundary layer.

The momentum equation is given by equation (12.21) except that now the Lorentz force is written as j B_y since (j) is now some known function of x.

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{dp}{dx} = \frac{\partial}{\partial y} \left(\eta \frac{\partial u}{\partial y} \right) + j B_y \qquad (12.45)$$

-23-

The energy equation written in terms of temperature ($h = c_p T$) is given by equation (12.25) except that here the Joule heating is written as j^2/σ again due to the fact that $j = j(x)$ is a given quantity. The continuity equation is, of course, equation (12.26).

Since (j) is the same within the boundary layer as in the free stream the momentum equation in the free stream is

$$\rho_\infty u_\infty \frac{du_\infty}{dx} + \frac{dp}{dx} = j\,B_y \tag{12.46}$$

and combining this with equation (12.45) gives

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\eta \frac{\partial u}{\partial y}\right) + \rho_\infty u_\infty \frac{du_\infty}{dx} \tag{12.47}$$

The energy equation evaluated in the free stream yields

$$\rho_\infty u_\infty c_p \frac{dT_\infty}{dx} = u_\infty \frac{dp}{dx} + \frac{j^2}{\sigma_\infty} \tag{12.48}$$

which when combined with equation (12.25) yields the following relation

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(K \frac{\partial T}{\partial y} + \frac{5kj}{2e} T\right) + \eta \left(\frac{\partial u}{\partial y}\right)^2$$
$$+ \rho_\infty u_\infty c_p \frac{dT_\infty}{dx} \frac{u}{u_\infty} + \frac{j^2}{\sigma_\infty}\left(\frac{\sigma_\infty}{\sigma} - \frac{u}{u_\infty}\right) \tag{12.49}$$

It is interesting to note at this point that the momentum equation for the present case is independent of B_y. The principle Magnetohydrodynamic effects appear in the energy equation.

For convenience in the boundary layer analysis μ and K will be assumed proportional to T, despite the fact that for constant mean free path, kinetic theory leads to a $T^{1/2}$ dependence. In general (see chapter 5) the electrical conductivity can be determined as a function of pressure and temperature. This time, for convenience in both the free stream and boundary layer analysis the pressure dependence will be ignored. Finally, the gas will be assumed perfect so that

$$p = \rho R T \tag{12.51}$$

Now, the boundary conditions which are needed are

$$T_\infty = T_\infty(x) \qquad\qquad u_\infty = u_\infty(x)$$

In addition, expressions are needed for $p(x)$ as well as $j(x)$. All four of these relations can be obtained from the solution of the inviscid problem. For the inviscid flow, it will be assumed that T_∞ and E_∞ are constant, the former also

leading to a constant σ_∞ if the pressure dependence of σ is neglected. Within these assumptions the solution, based on the methods of chapter 11, is

$$u_\infty = \alpha x^n$$

$$p = \frac{\sigma_\infty \, (R \, T_\infty \, E_\infty)^2}{\alpha^2} \quad \frac{(5n-1)}{n^2} \quad x^{1-5n} \tag{12.52}$$

$$j = \frac{\sigma_\infty \, R \, T_\infty \, E_\infty}{\alpha^2 n} \quad (5n-1) \; x^{-2n}$$

where one must have $n > 1/5$.

As noted earlier, u_∞ in the above form may make a similarity solution possible. The feasibility of reducing the equations to similar form will be taken up next. Define the following independent variables

$$\xi = \int_0^x \frac{p}{p_0} \; \frac{u_0}{u_\infty} \; dx \tag{12.53}$$

and

$$\zeta = \frac{u_\infty}{u_0} \sqrt{\frac{u_0}{2\nu_0}} \; \xi^{-1/2} \int_0^y \frac{\rho}{\rho_0} \; dy \tag{12.54}$$

where $(\;)_0$ denotes some convenient reference x position. Next, define a stream function in order that mass continuity be satisfied

$$\psi_y = \frac{\rho u}{\rho_0} \qquad\qquad \psi_x = - \frac{\rho v}{\rho_0}$$

and then redefine the stream function to be

$$\psi = \sqrt{2\nu_0 u_0} \; \sqrt{\xi} \quad f(\zeta)$$

The momentum equation then becomes

$$f''' + ff'' + \frac{2\xi}{u_\infty} \frac{du_\infty}{dx} \left[\frac{\rho_\infty}{\rho} - (f')^2 \right] = 0 \tag{12.55}$$

If a dimensionless temperature is defined as $\theta = \dfrac{T}{T_0}$ the energy equation becomes

$$\frac{1}{P_R} \frac{\partial^2 \theta}{\partial \zeta^2} + f \frac{\partial \theta}{\partial \zeta} = 2f'\xi \; \frac{\partial \theta}{\partial \xi} - (\gamma-1) \; M_\infty^2 \, (f'')^2$$

$$- \frac{5kj}{e} \left(\frac{u_0}{2\nu_0} \right)^{1/2} \frac{\sqrt{\xi}}{c_p \rho_0 u_\infty} \; \frac{p_0}{p} \frac{\partial \theta}{\partial \zeta}$$

$$- \frac{j^2}{\sigma} \quad \frac{2 u_o \xi}{c_p T_\infty \rho_\infty u_\infty^2} \quad \frac{p_o}{p} \quad \theta \left(\frac{\sigma_\infty}{\sigma} - f' \right) \tag{12.56}$$

Using the results of the inviscid analysis, equations (12.52), to determine ξ and restricting the value of n so that $n < \frac{1}{2}$ we find

$$\frac{x}{x_o} = \left[(2-4n) \left(\frac{\xi}{x_o} \right) \right]^{\frac{1}{2-4n}} \tag{12.57}$$

and the energy and momentum equations can be written as follows

$$\frac{1}{P_R} \frac{\partial^2 \theta}{\partial \eta^2} + f \frac{\partial \theta}{\partial \eta} = 2 f' \xi \frac{\partial \theta}{\partial \xi} - (\gamma-1) M_\infty^2 (f'')^2$$

$$- \left(\frac{(\gamma-1)}{\gamma} \right) \left(\frac{5n-1}{1-2n} \right)^{1/2} \frac{5 k T_o}{2e} \left(\frac{\sigma_o}{\eta_o R T_o} \right)^{1/2} \frac{\partial \theta}{\partial \eta}$$

$$- \left(\frac{\gamma-1}{\gamma} \right) \left(\frac{5n-1}{1-2n} \right) \theta \left(\frac{\sigma_\infty}{\sigma} - f' \right) \tag{12.58}$$

and

$$f''' + f f'' + \left(\frac{n}{1-2n} \right) \left[\theta - (f')^2 \right] = 0 \tag{12.59}$$

Accordingly, it can be seen that since $M_\infty = M_\infty (\xi)$ that the first two terms on the right hand side of equation (12.58) are functions of ξ and prevent a similar solution. If, however, the wall temperature is constant then $\theta \neq \theta (\xi)$, and the first of these terms vanishes. Finally, when $M_\infty^2 \cong 0$ the second term can be neglected and a similar solution is feasible if $M_\infty^2 \neq 0$ it can be taken to be a constant and the similarity is then in exact. In the present example one can clearly see that when the flow is compressible the specification that $u_\infty \alpha x^n$ is not sufficient to ensure a similar solution, but that additional conditions and assumptions are necessary.

Calculations based on the above equations have been carried out by Kerrobrock[6] for Helium seeded with Cesium. The free stream temperature and Mach number were taken to be 3000°K and unity respectively. A wall temperature of 1500°K was also chosen.

Within the boundary layer being considered heat is generated by two mechanisms: viscous dissipation, and Joule heating. Energy is transported from these two

sources and from the high temperature free stream to the wall by both conduction and diffusion of electrons against a temperature gradient. For the selection of parameters made by Kerrebrock both viscous dissipation and Joule heating are of comparable importance, and energy transport by electrons is only a few percent of the overall heat transfer.

To see more clearly the influence of the above heat sources on the boundary layer, velocity and temperature profiles are shown in Figures 12-11.
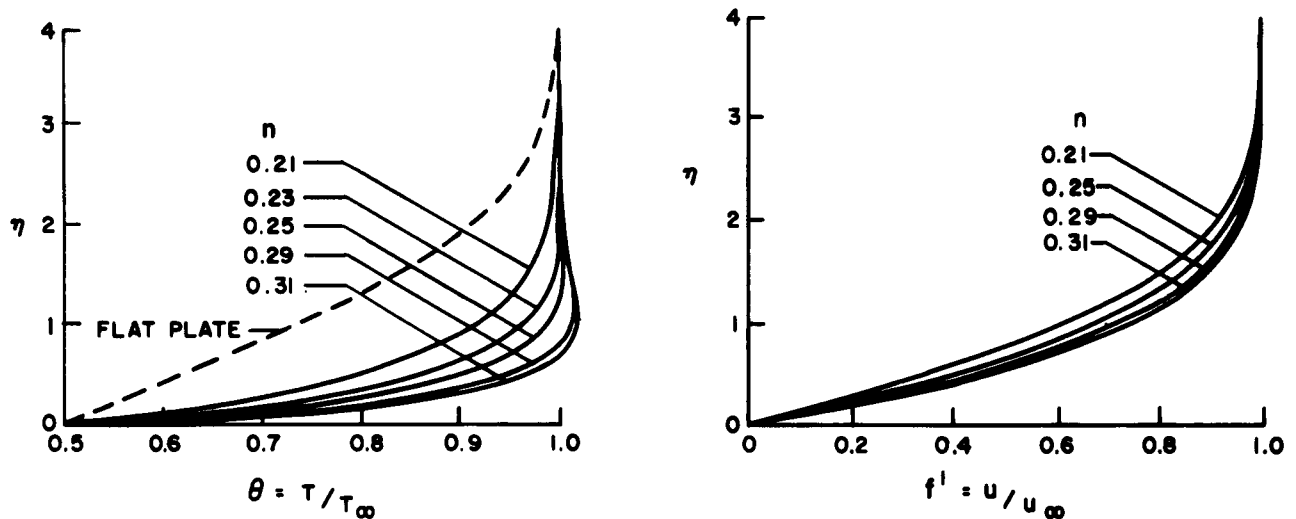

Figure 12-11. Temperature and Velocity Profiles for Electrode Boundary Layer. $M_\infty = 1$, $\theta_\omega = 0.5$

Increasing values of n correspond to free streams with increasing acceleration. This is seen to correspond to the fact that the velocity profiles tend to become fuller as n increases. The temperature profile marked "flat plate" corresponds to the case for which there is no current flowing at all. The large difference in $\frac{\partial \theta}{\partial \eta}\big|_o$ between the dashed curve and the others is indicative of the increased heat flux due to Joule heating in the low temperature region near the wall. For the numerical example chosen the increase is approximately an order of magnitude. The temperature excess at the highest accelerations can be attributed to heat generated by viscous dissipation which cannot readily transfer to the wall due to the large amount of heat liberated near the wall by Joule heating.

The final item of interest in regard to the electrode boundary layer is the electrode potential drop. Due to the fact that the gas in the vicinity of the wall is at low temperature and consequently low electrical conductivity, the electrical field in the vicinity of the wall will be much larger than that in the free stream in order to be consistent with a constant current flow. This will lead to a

larger potential drop across the boundary layer thickness than across a corresponding thickness of the free stream. Qualitatively, the potential distribution, based on the electric field seen by a stationary observer, across a complete channel will be as shown in Figure 12-12.
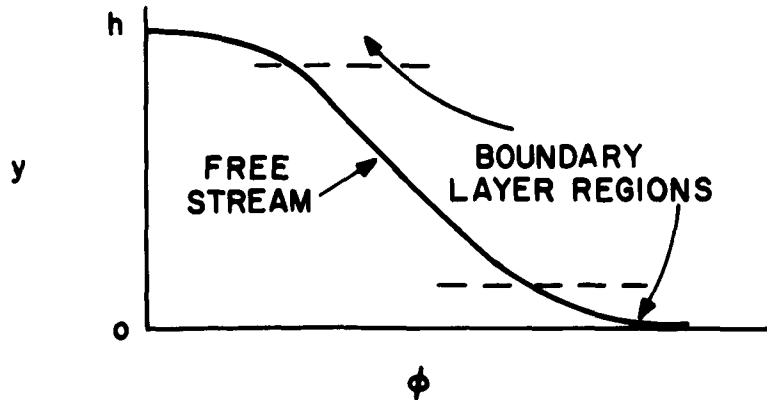

Figure 12-12. Potential Distribution Across a Channel of Height h.

The precise magnitude of the potential drop across the boundary layer can be calculated readily. Consider the potential drop in excess of the potential drop through an equivalent thickness of free stream. Then

$$\delta\phi = \int_0^\infty (E - E_\infty)\, dy = \int_0^\infty \left[\frac{j}{\sigma} - \frac{j}{\sigma_\infty} - B(u_\infty - u) \right] dy \qquad (12.60)$$

where it should be noted that the reduction in σ in the boundary layer will tend to increase $\delta\theta$ while the reduction in flow velocity there will tend to reduce it. Depending on the assumed conditions $\delta\theta$ can be positive or negative. Calculations of $\delta\theta$ based on the example cited earlier are shown in Figure 12-13.
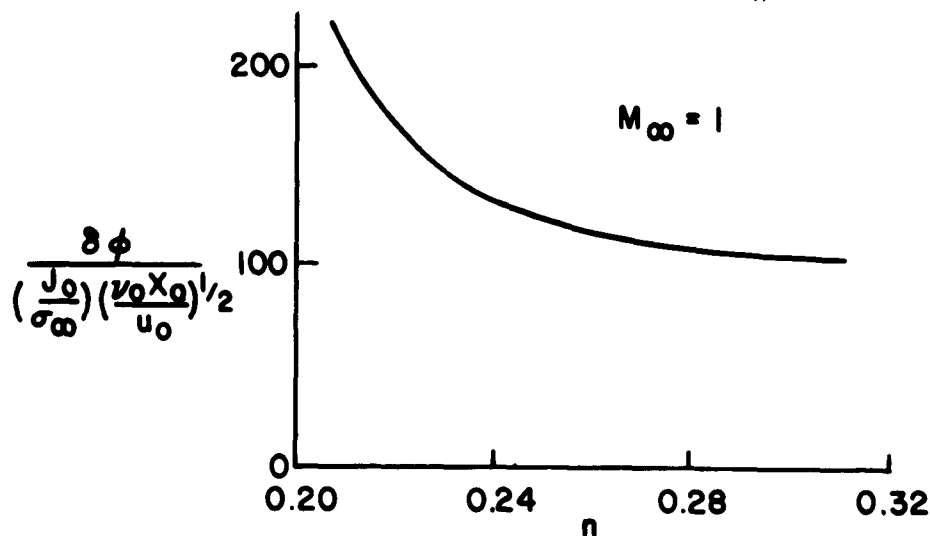

Figure 12-13. Boundary Layer Potential Excess

It is interesting to observe that $\delta\phi$ is positive, for this case, so that the potential drop through the boundary layer is indeed greater than in the free stream over the same distance.

Finally, mention should be made of the magnetohydrodynamic compressible boundary layer in the region of the stagnation point on a blunt body. The practical example is, of course, the re-entering nose cone. For this problem the inviscid flow has been calculated and it has been shown that $u_\infty = \alpha x$ when B_y is a constant, the magnitude of α being reduced as the strength of the magnetic field is increased. These are precisely the minimum requirements for a similar solution, as noted earlier, and with several additional simplifying assumptions similarity solutions can indeed be obtained.[7]

12.6 MAGNETIC BOUNDARY LAYERS

Up to this point in the present chapter, and in fact throughout this book, problems of external flow (flow over closed bodies such as airfoils) have not been considered. It will be of interest, however, to consider qualitatively some new boundary layer phenomena that arise in such flows when R_m is large and P_{R_m} is small.[8]

The particular overall problem that will be investigated will be the so called "alligned flows". These are flows in which the flow velocity and magnetic field vectors far from the body are parallel. Now when $R_m = \infty$ and the electric field, $\underline{E}$, is zero the Ohm's law requires that $\underline{v} \times \underline{B} = 0$ so that $\underline{v}$ and $\underline{B}$ are not only parallel at infinity but are parallel everywhere. For a body of finite conductivity $\underline{E} = 0$ implies $\underline{j} = 0$ so that the magnetic field is harmonic within the body. Since it must be a constant on the surface the mean value theorem of potential theory tells us that it must be zero everywhere within the body. Accordingly, the tangential component of $\underline{B}$ at the surface must jump from a finite value to zero. As was shown in Chapter 2 this corresponds to a current sheet at the surface. Since $\underline{v}$ must also go to zero at the surface there must also be a vortex sheet there. When R_m is large but no longer infinite these current and vortex sheets are in reality boundary layers. The physical nature of such layers, and the equations governing them will be the subject of this section.

Since, as was seen earlier, $P_{R_m} = \dfrac{R_m}{Re}$ the assumption of large R_m and small P_{R_m} is tantamount to assuming R_m large and R_e much larger. Accordingly, it can be anticipated that the boundary layer in question will really be two layers. One will be an outer layer in which viscosity is negligible, and the other will be a viscous sublayer. The outer layer thickness will be the order of $R_m^{-1/2}$,

while the inner layer will be the order of $R_e^{-1/2}$. The flow external to the
outer layer, where $\underline{v} \, \alpha \, \underline{B}$, will be irrotational. The above physical interpretation
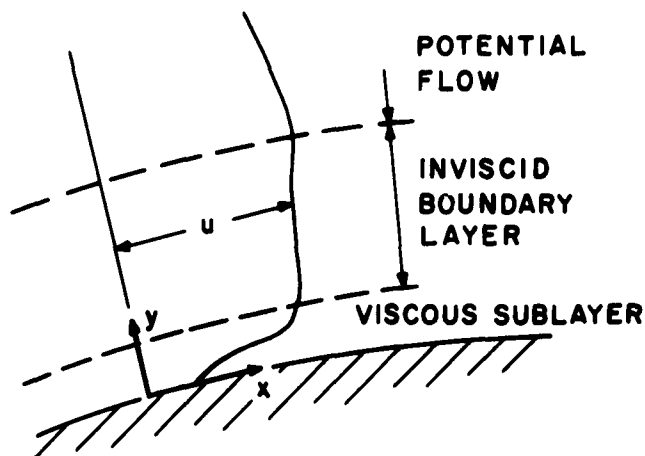of the boundary layer region is illustrated in Figure 12-14.

Figure 12-14.   Sketch Showing Inviscid Magnetohydrodynamic
Boundary Layer and Viscous Sublayer

Before considering the equations and specific boundary conditions in detail,
it will be of value to first discuss the procedure for solution.   First, the
potential flow must be determined neglecting both boundary layers.   The boundary
conditions for such a solution are that the surface be both a fluid and magnetic
streamline.   From such a solution one obtains values of u and B_x at the wall.
These, then, will serve as outer boundary conditions for the inviscid layer.
The inner conditions for this layer are evaluated by assuming the viscous layer
to be of negligible thickness.   Thus, one of the inner conditions on the inviscid
layer will be v = 0 while u $\neq$ 0.   Another inner boundary condition on the magnetic
field will be needed to complete the formulation of the inviscid boundary layer.
The viscous layer will then use u (x, o) as obtained from the inviscid layer solu-
tion as its outer boundary condition along with v = 0.   The inner boundary
conditions on the viscous layer are the conventional ones of u = v = 0.

So far little has been said about the boundary conditions on $\overline{B}$.   The
difficulty lies in the fact that when R_m is no longer infinite then $\underline{B}$ is no longer
zero within the body and B_x and B_y at the surface are not known prior to the
solution of the problem.   The one thing we do know, however, is that when
$R_m \rightarrow \infty$ then B_x and B_y should both $\rightarrow$ 0 within the body.

Before resolving the above question of the boundary values of B_x and B_y it will be necessary to introduce some order of magnitude arguments. It will be assumed that the thickness of the inviscid magnetic boundary layer, δ_i, is $\mathcal{O}(R_m^{-1/2})$, and that the thickness of the viscous sublayer, δ_v, is $\mathcal{O}(R_e^{-1/2})$. Also, differentiation with respect to x will not alter the order of magnitude of a quantity, while differentiation with respect to y will change the order of magnitude by δ^{-1}. Whether it is δ_i or δ_v will depend on which layer is being discussed.

It was noted earlier that B_x and B_y tend to zero as $R_m \rightarrow \infty$, so that it will be valid to assume them both $\mathcal{O}(R_m^{-1/2})$ at the wall. Since the viscous sublayer is extremely thin neither should vary appreciably from its value at the wall within this region. Next, consider the following relation

$$\underline{\nabla} \times \underline{B} = R_m \ \underline{v} \times \underline{B} \tag{12.61}$$

where $\underline{v}$ and $\underline{B}$ have been made dimensionless by reference values of v and B, where Ohm's law has been used, and $\underline{E} = 0$. Or, for the present two dimensional problem:

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = R_m \left[u\,B_y - v\,B_x \right] \tag{12.62}$$

Within the viscous layer the first two terms have the following order of magnitudes

$$\frac{\partial B_y}{\partial x} \rightarrow \mathcal{O}(R_m^{-1/2}) \qquad \frac{\partial B_x}{\partial y} = \mathcal{O}(P_{R_m}^{-1/2})$$

so that $\partial B_y / \partial x \ll \dfrac{\partial B_x}{\partial y}$. Also, within the viscous layer $v \ll u$ and B_y and B_x are of comparable order of magnitudes, so that $v\,B_x \ll u\,B_y$. Accordingly equation (12.61) can be simplified, within the viscous layer to

$$- \frac{\partial B_x}{\partial y} = R_m\, u\, B_y \tag{12.63}$$

Or, integrating over the viscous layer

$$B_x = B_x(x, o) + \mathcal{O}(R_m^{1/2}\ R_e^{-1/2})$$

$$B_x = O(R_m^{-1/2}) + \mathcal{O}(P_{R_m}^{1/2})$$

Accordingly, one can assume $B_x \cong 0$ throughout the viscous sublayer and take this as the other inner boundary condition for the inviscid layer.

The equations governing both boundary layers will be deduced next, by similar order of magnitude arguments. Within the <u>inviscid</u> layer $\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$, so that assuming $B_x = \mathcal{O}(1)$ leads to $B_y = \mathcal{O}(R_m^{-1/2})$. Next, the two momentum equations (equation 8.17), and equation (12.62) are considered

$$\overset{\mathcal{O}(1)}{} \qquad \overset{\mathcal{O}(1)}{} \qquad \overset{\mathcal{O}(1)}{} \qquad \overset{\mathcal{O}(1)}{} \qquad \overset{\mathcal{O}(1)}{}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{\partial P}{\partial x} = N \left\{ B_x \frac{\partial B_x}{\partial x} + B_y \frac{\partial B_x}{\partial y} \right\} \tag{12.64}$$

$$\overset{\mathcal{O}(R_m^{-1/2})}{} \quad \overset{\mathcal{O}(R_m^{-1/2})}{} \qquad \overset{\mathcal{O}(R_m^{-1/2})}{} \quad \overset{\mathcal{O}(R_m^{-1/2})}{}$$

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \frac{\partial P}{\partial y} = N \left\{ B_x \frac{\partial B_y}{\partial x} + B_y \frac{\partial B_y}{\partial y} \right\} \tag{12.65}$$

$$\overset{\mathcal{O}(R_m^{-1/2})}{} \quad \overset{\mathcal{O}(R_m^{1/2})}{} \quad \overset{\mathcal{O}(R_m^{-1/2})}{} \quad \overset{\mathcal{O}(R_m^{-1/2})}{}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = R_m \left\{ u B_y - v B_x \right\} \tag{12.66}$$

where $P = p + \frac{B^2}{2}$ and $N = \frac{B_\infty^2}{\mu \rho u_\infty^2}$ and it will be assumed that P and N are $\mathcal{O}(1)$. The resulting boundary layer equations in dimensional form for the <u>inviscid</u> layer are

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{\partial P}{\partial x} = B_x \frac{\partial B_x}{\partial x} + B_y \frac{\partial B_x}{\partial y} \tag{12.67}$$

$$- \frac{\partial B_x}{\partial y} = \sigma \mu_e (u B_y - v B_x) \tag{12.68}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{12.69}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \tag{12.70}$$

with the boundary conditions

at y = ∞ $u = u_\infty(x)$

$B_x = B_{x_\infty}(x) \quad \alpha \; u_\infty(x)$

at y = 0 $v = 0$

$B_x = 0$

It is interesting to see that a boundary condition on B_y at y = 0 is not needed since when B_y is eliminated from the equations by equation (12.70) they become second order in B_x and there are two boundary conditions on B_x available. The value of B_y at y = 0 is part of the solution being sought and, accordingly, it is not a suitable boundary condition. From equation (12.65) it can also be seen that $\frac{\partial P}{\partial y} = \mathcal{O}(R_m^{-1/2})$ and P = P(x) alone. Accordingly, $\frac{\partial P}{\partial x}$ can be evaluated from the free stream solution.

The boundary layer equations within the viscous layer are precisely those obtained earlier in the present chapter. The applied magnetic field, B_y, is taken to be the value found at y = 0 from the inviscid boundary layer solution. The pressure, p, to be used is P(x) obtained from the potential flow solution.

In conclusion then, when $R_m \to \infty$ boundary layer phenomena very similar to conventional boundary layers can exist. They do, however, offer a number of new features for future studies.[8]

REFERENCES

1. Ludford, G.S.S.   Rayliegh's Problem in Hydromagnetics: The Impulsive Motion of a Pole Piece. Archiv for Rational Mechanics and Analysis, $\underline{3}$, 14 (1959).

2. Rossow, V.J.   On Flow of Electrically Conducting Fluids Over a Flat Plate in the Presence of a Transverse Magnetic Field. NASA Report 1358, 1958

3. Sherman, A.   Viscous Magnetohydrodynamic Boundary Layer. Phys. Fluids $\underline{4}$, 552 (1961)

4. Howarth, L. Ed.   Modern Developments in Fluid Dynamics. High Speed Flow, vol. 1 Oxford University Press, 1953.

5. Bush, W.B.   Compressible Flat Plate Boundary Layer Flow with an Applied Magnetic Field. J. Aerospace Sci. $\underline{27}$, 49 (1960)

6. Kerrebrock, J.L.   Electrode Boundary Layers in Direct Current Plasma Accelerators. J. Aerospace Sci. $\underline{28}$, 631 (1961).

7. Bush, W.B.   The Stagnation-Point Boundary Layer in the Presence of an Applied Magnetic Field. J. Aerospace Sci. $\underline{28}$, 610 (1961)

8. Sears, W.R.   On a Boundary Layer Phenomenon in Magneto-Fluid Dynamics. Astronautica Acta $\underline{7}$, 223 (1961).

GENERAL ⬤ ELECTRIC

SPACE SCIENCES LABORATORY
MISSILE AND SPACE DIVISION

TECHNICAL INFORMATION SERIES

AUTHOR	SUBJECT CLASSIFICATION	NO.
		R63SD4
A. Sherman	Magnetohydrodynamics	DATE
		Jan. 7, 1963

TITLE	G. E. CLASS
MAGNETOHYDRODYNAMIC	I
BOUNDARY LAYER	GOV. CLASS
	None

REPRODUCIBLE COPY FILED AT M&D LIBRARY, DOCUMENTS LIBRARY UNIT, VALLEY FORGE SPACE TECHNOLOGY CENTER, KING OF PRUSSIA, PA.	NO. PAGES
	37

SUMMARY

A review of those new phenomena which arise in magnetohydrodynamic boundary layers is presented in this report. The first topic discussed due to its simplicity and relation to boundary layer flows is the Rayleigh problem. Here it is shown that the introduction of magnetohydrodynamic forces leads to basic changes in the nature of the flow. Next the basic boundary layer equations are deduced and the conditions necessary to yield similar solutions deduced. In succeeding sections incompressible and compressible boundary layers are treated with attention paid to the appropriate external boundary conditions. Finally, boundary layer phenomena arising when $R_m \to \infty$ are considered in order to illustrate some new phenomena within the framework of boundary layer theory. It is intended that the present material will be one chapter of a forthcoming book.

By cutting out this rectangle and folding on the center line, the above information can be fitted into a standard card file.

AUTHOR _Arthur Sherman_

COUNTERSIGNED _W. M. Sutton_   _Joseph Farber_